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On the Linearity of Free Nilpotent-by-Abelian Groups*

C. K. GUPTA AND N. D. GUPTA

Department of Mathematics, University of Manitoba, Winnipeg, Canada

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INTRODUCTION

Linearity of free nilpotent groups seems to be well known. Among the other relatively free linear groups are: $F/\delta_2(F)$ (W. Magnus [6]), $F/\gamma_2\gamma_3(F) \cdot \gamma_3\gamma_2(F)$ (C. K. Gupta [3]) and $F_3/[\delta_2(F_3), F_3]$ (C. K. Gupta [2]). In this paper we show that $F/\gamma_n\gamma_2(F)$ $(n \geqslant 2)$ is isomorphic to a linear group. This answers in the affirmative a recent question of B.A.F. Wehrfritz raised in Oberwolfach, May 1971.

NOTATION

Most of our notation will be developed as we proceed. However, the following notation is standard: $[a, b] = a^{-1}b^{-1}ab$; $[a, b, c] = [[a, b], c]$; $\gamma_m(A)$ is the m-th term of the lower central series of A; $\delta_m(A)$ is the m-th term of the derived series of A; $\gamma_r\gamma_s(A) = \gamma_r(\gamma_s(A))$. The free group of countable infinite rank or finite (unspecified) rank is denoted by F, whereas F_m denotes the free group of rank m.

PRELIMINARIES

Let $F_m (m \geqslant 2)$ be the free group of rank m freely generated by $a_1, \ldots, a_m$. Modulo $F_m'(=\gamma_2(F_m))$ define an order relation "$<$" on the elements of F_m as follows:

$$u = a_1^{\alpha_1} \cdots a_m^{\alpha_m} < v = a_1^{\beta_1} \cdots a_m^{\beta_m} \qquad (\alpha_i, \beta_j \in Z)$$

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if

(i) $\sum_{i=1}^{m} |\alpha_i| < \sum_{i=1}^{m} |\beta_i|$; or

(ii) $\sum_{i=1}^{m} |\alpha_i| = \sum_{i=1}^{m} |\beta_i|$, and there is a least integer $t \in \{1,..., m\}$ such that $|\alpha_i| = |\beta_i|$ for $i < t$ and $|\alpha_t| < |\beta_t|$; or

(iii) $|\alpha_i| = |\beta_i|$ for all $i \in \{1,..., m\}$ and there is a least integer $t \in \{1,..., m\}$ such that $\alpha_i = \beta_i$ for $i < t$ and $\alpha_t < \beta_t$.

For each pair (i, j), $m \geqslant i > j \geqslant 1$, put

$$c_{ij}(k) = [a_i, a_j]^{u_{ij}(k)},$$

where $u_{ij}(k) = a_j^{\alpha_{jk}} \cdots a_m^{\alpha_{mk}}$ for some $\alpha_{jk},..., \alpha_{mk} \in Z$.

We extend the order relation "$<$" to $c_{ij}(k)\,s'$ as follows: define

$$c_{ij}(k) < c_{i'j'}(k')$$

if

(i) $u_{ij}(k) < u_{i'j'}(k')$; or

(ii) $u_{ij}(k) = u_{i'j'}(k')$ and $i < i'$; or

(iii) $u_{ij}(k) = u_{i'j'}(k')$, $i = i'$ and $j < j'$.

Let $C = \{c_{ij}(k), m \geqslant i > j \geqslant 1, k = 1, 2,...\}$. Using the above order relation we may re-arrange the contents of C as

$$B_1(=C) = \{b_{11}, b_{12},..., \mid b_{1i} < b_{1j} \text{ for } i < j\}.$$

Using B_1 as an ordered set of symbols we let B_n denote the inductively defined and ordered set of (standard) basic commutators of weight $n (n \geqslant 1)$, B_1 being the ordered set of basic commutators of weight 1 (see, for instance, M. Hall [5, Chapter 11]). Thus for each $n \geqslant 1$ we may write

$$B_n = \{b_{n1}, b_{n2},...\}, \qquad \text{where} \qquad b_{n1} < b_{n2} < \cdots.$$

We now prove,

LEMMA 1. *Modulo $\gamma_{n+1}(F_m')$, B_n generates $\gamma_n(F_m')$.*

Proof. Modulo $\gamma_2(F_m')$, F_m' is clearly generated by all elements of the form $[a_i, a_j]^w$ $(i > j)$, where $w = a_1^{\alpha_1} \cdots a_m^{\alpha_m}$. Using the standard identities

$$[a_i^{\epsilon_i}, a_j^{\epsilon_j}] = [a_i, a_j]^{\epsilon_i \epsilon_j a_i^{(\epsilon_i - 1)/2} a_j^{(\epsilon_j - 1)/2}}, [a, b, c] = [a, c, b][b, c, a]^{-1}$$

modulo $\gamma_2(F_m')$ and $[a, b]^c = [a, b][a, b, c] (\epsilon_i, \epsilon_j = \pm 1)$, it follows that $[a_i, a_j]^{a_k^{\epsilon_k}}$ $(i > j > k) \in gp\{B_1\}$ modulo $\gamma_2(F_m')$. Thus by repeated application of this last observation it follows that $[a_i, a_j]^w$ $(i > j) \in gp\{B_1\}$ for all $w \in F_m$. This proves the lemma for $n = 1$. Now modulo $\gamma_{n+1}(F_m')$, $\gamma_n(F_m')$ is generated by all left-normed commutators of weight n whose entries are the generators of F_m', namely, b_{1k}'s. In other words, $\gamma_n(F_m')$ is generated by all commutators of the form

$$[b_{1i_1}, \ldots, b_{1i_n}].$$

But $[b_{1i_1} \ldots, b_{1i_n}] \in gp\{B_n\}$ and we have the required proof.

Remark 1. As an application of our main result we shall also show that $\gamma_n(F_m')$ is freely generated by B_n modulo $\gamma_{n+1}(F_m')$ for $n \geqslant 1$ (c.f. Bachmuth [1] for the case $n = 1$).

Let H be the free *abelian* group of rank $2m(m \geqslant 2)$ freely generated by $y_1, \ldots, y_m, z_1, \ldots, z_m$ and let $P = \{\rho_1, \rho_2, \ldots\}$ be a set of independent and commuting indeterminates. Consider $S = ZH[P]$ to be the polynomial ring in ρ's over the integral group ring ZH. For each pair (i, j), $m \geqslant i > j \geqslant 1$, define

$$\mu_{ij} = y_i^{-1} z_i^{-1}(-1 + y_j)\, \rho_i - y_j^{-1} z_j^{-1}(-1 + y_i)\, \rho_j .$$

We prove

LEMMA 2. *In* S, *suppose* $\sum_{(i,j)=(2,1)}^{(m,m-1)} (\sum_{k=1}^{t_{ij}} \zeta_{ij}(k)\, w_{ij}(k)\, \mu_{ij}) = 0$, *where*

(i) $\{\zeta_{ij}(k), k = 1, \ldots, t_{ij}\}$ *is a set of elements of* S *such that each term of* $\zeta_{ij}(k)$ *is of degree zero in each* $y_1, \ldots, y_m, \rho_1, \ldots, \rho_m$; *and*

(ii) $\{w_{ij}(k), k = 1, \ldots, t_{ij}\}$ *is a set of words in* y's *of the form* $y_j^{\alpha_{jk}} \cdots y_m^{\alpha_{mk}}$ *with* $w_{ij}(1) < \cdots < w_{ij}(t_{ij})$ (*the ordering being parallel to that of* F_m/F_m'). *Then* $\zeta_{ij}(k) = 0$ *for all* $(i, j)(m \geqslant i > j \geqslant 1)$ *and all* $k \in \{1, \ldots, t_{ij}\}$.

Proof. Put $\tau_{ij} = \sum_{k=1}^{t_{ij}} \zeta_{ij}(k)\, w_{ij}(k)$. Then we first of all prove by induction on $m \geqslant 2$, that $\tau_{ij}\mu_{ij} = 0$ for all (i, j), $m \geqslant i > j \geqslant 1$. If $m = 2$, there is nothing to prove. Let $m > 2$ and assume that the result holds for $m - 1$. We have

$$0 = \sum_{(i,j)=(2,1)}^{(m,m-1)} \tau_{ij}\mu_{ij}$$

$$= \sum_{j=1}^{m-1} \tau_{mj}\mu_{mj} + \sum_{(i,j)=(2,1)}^{(m-1,m-2)} \tau_{ij}\mu_{ij} .$$

Thus by the induction hypothesis it is enough to show that $\tau_{mj}\mu_{mj} = 0$ for

$j = 1, \ldots, m-1$. Since ρ's are independent, the sum of the coefficients of ρ_m in $\sum_{(i,j)=(2,1)}^{(m,m-1)} \tau_{ij}\mu_{ij}$ must be zero. Accordingly, we have

$$\begin{aligned} 0 &= \sum_{j=1}^{m-1} \tau_{mj}(y_m^{-1} z_m^{-1}(-1+y_j)) \\ &= (y_m^{-1} z_m^{-1}) \sum_{j=1}^{m-1} \tau_{mj}(-1+y_j) \\ &= \sum_{j=1}^{m-1} \tau_{mj}(-1+y_j) \\ &= \tau_{m1}(-1+y_1) + \xi, \end{aligned}$$

where each term in $\xi = \sum_{j=2}^{m-1} \tau_{mj}(-1+y_j)$ is of degree zero in y_1 (by definition). Writing

$$\tau_{m1} = \sum_{i=-t}^{+t} \xi_i y_1{}^i,$$

where $\xi_i s'$ are all of degree zero in y_1, we have

$$0 = \sum_{i=-t}^{+t} \xi_i y_1{}^i(-1+y_1) + \xi.$$

Equating the coefficients of $y_1{}^i$ to 0 for all $i = -t, \ldots, t+1$, we get

$$\begin{aligned} 0 = \xi_{-t} = \xi_{-t} - \xi_{-t+1} = \cdots = \xi_{-2} - \xi_{-1} = \xi_{-1} - \xi_0 + \xi = \xi_0 - \xi_1 \\ = \cdots = \xi_{+t-1} - \xi_t = \xi_t . \end{aligned}$$

Solving all these equations we see that

$$0 = \xi_{-t} = \cdots = \xi_{+t} = \xi,$$

and in particular $\tau_{m1} = 0$. Similarly $0 = \tau_{m2} = \cdots = \tau_{m,m-1}$. Thus we have shown that $\tau_{ij}\mu_{ij} = 0$ for $m \geqslant i > j \geqslant 1$.

Next we consider.

$$\begin{aligned} 0 &= \tau_{ij}\mu_{ij} \\ &= \sum_{k=1}^{t_{ij}} \zeta_{ij}(k)\, w_{ij}(k)\, \mu_{ij} . \end{aligned}$$

Since the sum of the coefficients of ρ_i in $\tau_{ij}\mu_{ij}$ is zero, we have

$$0 = \left(\sum_{k=1}^{t_{ij}} \zeta_{ij}(k)\, w_{ij}(k)\right) (y_i^{-1} z_i^{-1}(-1 + y_j))$$

$$= \sum_{k=1}^{t_{ij}} \zeta_{ij}(k)\, w_{ij}(k), \text{ since } S \text{ is an integral domain.}$$

But $\zeta_{ij}(k)$'s are all of degree zero in y's and $w_{ij}(k)$'s are all distinct words in y's. Thus $\sum_{k=1}^{t_{ij}} \zeta_{ij}(k)\, w_{ij}(k) = 0$ implies that $\zeta_{ij}(k)\, w_{ij}(k) = 0$ and hence $\zeta_{ij}(k) = 0$ for all $k = 1,\ldots, t_{ij}$. This completes the proof of the lemma.

As an immediate corollary to Lemma 2 we record,

LEMMA 3. *The conclusion of Lemma 2 remains valid if μ_{ij} is replaced by $\mu_{ij}^* = y_i^{-1}(-1 + y_j z_j^{-1})\,\rho_i - y_j^{-1}(-1 + y_i z_i^{-1})\,\rho_j$ and $w_{ij}(k) = y_j^{\alpha_{jk}} \cdots y_m^{\alpha_{mk}}$ is replaced by $w_{ij}^*(k) = (y_j^{\alpha_{jk}} \cdots y_m^{\alpha_{mk}})(z_j^{-\alpha_{jk}} \cdots z_m^{-\alpha_{mk}})$.*

Proof. Since H is freely generated by $y_1,\ldots, y_m, z_1,\ldots, z_m$ it is also freely generated by $y_1 z_1^{-1},\ldots, y_m z_m^{-1}, z_1,\ldots, z_m$. Further since μ_{ij}^* and $w_{ij}^*(k)$ are obtained from μ_{ij} and $w_{ij}(k)$, respectively, by replacing each y_i by $y_i z_i^{-1}$ and $\zeta_{ij}(k)$'s are unaffected, we have the required proof.

THE MAIN RESULT

Let G be the free abelian group freely generated by

$$\{x_{ij},\, i = 1,\ldots, n + 1 (n \geqslant 1), j = 1, 2,\ldots\}$$

and let $\Lambda = \{\lambda_{i,i-1}^{(k)},\, n + 1 \geqslant i \geqslant 2, k = 1, 2,\ldots\}$ be a set of independent and commuting indeterminates. Consider the polynomial ring $R = ZG[\Lambda]$ over the group ring ZG. Let M_{n+1} denote the multiplicative group of $n + 1 \times n + 1$ matrices over R generated by $\langle a_1\rangle, \langle a_2\rangle,\ldots,$ where

$$\langle a_k\rangle = \begin{bmatrix} x_{1k} & & & & \bigcirc \\ \lambda_{21}^{(k)} & x_{2k} & & & \\ 0 & \lambda_{32}^{(k)} & x_{3k} & & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & \cdot & \cdot & 0 \quad \lambda_{n+1,n}^{(k)} & x_{n+1,k} \end{bmatrix}$$

for $k = 1, 2, \ldots$. Let F be the free group freely generated by $a_1, a_2, \ldots$. If $w = a_{i_1}^{\epsilon_1} \cdots a_{i_l}^{\epsilon_l} (\epsilon_i \in \{-1, 1\})$ is an arbitrary word in F, we define

$$\langle w \rangle = \langle a_{i_1} \rangle^{\epsilon_1} \cdots \langle a_{i_l} \rangle^{\epsilon_l},$$

so that the mapping

$$\varphi \colon w \to \langle w \rangle$$

defines a homomorphism of F onto M_{n+1}. Our main result may now be stated as

THEOREM. *The kernel of φ is equal to $\gamma_{n+1}(\gamma_2(F))$.*

To simplify calculations we introduce certain mappings

$$\alpha_{ij} (n + 1 \geqslant i \geqslant j \geqslant 1)$$

of F into R defined as:

$$\alpha_{ij}(w) = ij\text{-entry of the matrix } \langle w \rangle, \text{ for all words } w \in F.$$

Further, to each $w = a_{i_1}^{\epsilon_1} \cdots a_{i_l}^{\epsilon_l}$ in F we define $\mathbf{w}(j) = x_{ji_1}^{\epsilon_1} \cdots x_{ji_l}^{\epsilon_l} \in G$ for $j = 1, \ldots, n + 1$. In particular, $\mathbf{w}(j) = 1$ if and only if $w \in F' (= \gamma_2(F))$. We record some of the consequences of our definitions in the following lemma.

LEMMA 4. *Let $u, v, w, \ldots$ be words in F, then*

(i) $\alpha_{ii}(u) = \mathbf{u}(i)$;

(ii) $\alpha_{ij}(e) = 0$ *for* $i > j$ (*e is the identity of F*);

(iii) $\alpha_{ij}(uv) = \sum_{k=j}^{i} \alpha_{ik}(u)\, \alpha_{kj}(v)$;

(iv) $\alpha_{i,i-1}(a_k) = \lambda_{i,i-1}^{(k)}$;

(v) $\alpha_{i,i-1}(u^{-1}) = -\mathbf{u}_{(i-1)}^{-1} \mathbf{u}_{(i)}^{-1} \alpha_{i,i-1}(u)$;

(vi) $$\alpha_{i,i-1}[u, v] = \mathbf{u}_{(i)}^{-1}(-1 + \mathbf{v}_{(i-1)} \mathbf{v}_{(i)}^{-1})\, \alpha_{i,i-1}(u) - \mathbf{v}_{(i)}^{-1}(-1 + \mathbf{u}_{(i-1)} \mathbf{u}_{(i)}^{-1})\, \alpha_{i,i-1}(v);$$

(vii) $\alpha_{i,i-1}[u, v]^w = \mathbf{w}_{(i-1)} \mathbf{w}_{(i)}^{-1} \alpha_{i,i-1}[u, v]$;

(viii) *if $u \in \gamma_r \gamma_2(F)$ then $\alpha_{ij}(u) = 0$ for all $r > i - j$*;

(ix) *if $u \in \gamma_r \gamma_2(F)$, $v \in \gamma_s \gamma_2(F)$ and $r + s = i - j$, then*
$$\alpha_{ij}[u, v] = \alpha_{i,i-r}(u)\, \alpha_{i-r,j}(v) - \alpha_{i,i-s}(v)\, \alpha_{i-s,j}(u)$$

(x) *if $u, v \in \gamma_r \gamma_2(F)$ and $r = i - j$, then*
$$\alpha_{ij}(u^l) = l\alpha_{ij}(u) \qquad \textit{and} \qquad \alpha_{ij}(uv) = \alpha_{ij}(u) + \alpha_{ij}(v).$$

Proof. The proofs of (i)–(iv) are trivial. For the proof of (v) note by (i)–(iv) that

$$\begin{aligned} 0 &= \alpha_{i,i-1}(e) = \alpha_{i,i-1}(uu^{-1}) \\ &= \alpha_{i,i-1}(u)\,\alpha_{i-1,i-1}(u^{-1}) + \alpha_{i,i}(u)\,\alpha_{i,i-1}(u^{-1}) \\ &= \mathbf{u}_{(i-1)}^{-1}\alpha_{i,i-1}(u) + \mathbf{u}_{(i)}\alpha_{i,i-1}(u^{-1}). \end{aligned}$$

The proofs of (vi) and (vii) can be derived by repeated applications of (iii) and (v). For the proof of (viii) we refer the reader to formula (6) of Gupta–Gupta [4]. Lemma 3(b) of [4] together with (viii) proves (ix). Finally (x) follows by the product rule (iii) and (viii).

For $i-j \in \{1,..., n\}$ let $(R_{i-j}, +)$ denote the additive subgroup of R generated by all products of the form

$$\alpha_{i,i-1}(b_{1k_1}) \cdots \alpha_{j+1,j}(b_{1k_{i-j}}),$$

where b_{1k}'s are from B_1. By Lemma 4(vi), (vii) we can identify each $\alpha_{r,r-1}(b_{1k})$ with an element of the form $w_{ij}^*(k)\,\mu_{ij}^*$ (as given in Lemma 3). Further we note that if $r \neq s$ then $\alpha_{r,r-1}(b_{1k})$ and $\alpha_{s,s-1}(b_{1k'})$ involve independent symbols. Thus by repeated applications of Lemma 3 we have the following important lemma.

LEMMA 5. *$(R_{i-j}, +)$ is freely generated by products of the form $\alpha_{i,i-1}(b_{1k_1}) \cdots \alpha_{j+1,j}(b_{1k_{i-j}})$.*

Let N denote the free associative ring generated by $\nu_1, \nu_2,...$ and let $(N_{i-j}, +)$, $i-j \in \{1,..., n\}$ denote the additive subgroup of N generated by all elements of the form

$$\nu_{k_1} \cdots \nu_{k_{i-j}}.$$

By Lemma 5, the mapping

$$\chi_{ij} : \nu_{k_1} \cdots \nu_{k_{i-j}} \to \alpha_{i,i-1}(b_{1k_1}) \cdots \alpha_{j+1,j}(b_{1k_{i-j}})$$

defines an isomorphism of $(N_{i-j}, +)$ onto $(R_{i-j}, +)$. By Lemma 4(ix), it is easy to see that if

$$\beta(b_{1k_1},..., b_{1k_{i-j}}) \in B_{i-j}$$

for some bracket arrangement "β", then

$$\alpha_{ij}(\beta(b_{1k_1},..., b_{1k_{i-j}}))\,\chi_{ij}^{-1} = \beta(\nu_{k_1},..., \nu_{k_{i-j}})$$

is a basic Lie-element in N with the same bracket arrangement "β", in which each occurrence of b_{1k} has been replaced by ν_k. Thus if

$$\sum_{k=1}^{t} r_k \alpha_{ij}(b_{i-j,k}) = 0,$$

then we have in turn

$$\left(\sum_{k=1}^{t} r_k \alpha_{ij}(b_{i-j,k})\right) \chi_{ij}^{-1} = 0;$$

$$\sum_{k=1}^{t} r_k \alpha_{ij}(\beta_k(b_{1k_1},\ldots, b_{1k_{i-j}}))\, \chi_{ij}^{-1} = 0;$$

$$\sum_{k=1}^{t} r_k \beta_k(\nu_{k_1},\ldots, \nu_{k_{i-j}}) = 0;$$

which implies that $r_k = 0$ for $k = 1,\ldots, t$ since basic Lie-elements in N are linearly independent (M. Hall [5, Chapter 11]). We record this observation as

LEMMA 6. $\sum_{k=1}^{t} r_k \alpha_{ij}(b_{i-j,k}) = 0$ *implies* $r_k = 0$ *for each* $k = 1,\ldots, t$.

PROOF OF THE MAIN THEOREM

By Lemma 4(viii),

$$\alpha_{ij}(u) = 0 \qquad \text{for all} \quad u \in \gamma_{n+1}\gamma_2(F)$$

and all (i, j), $n + 1 \geqslant i > j \geqslant 1$. Thus $\gamma_{n+1}\gamma_2(F) \subseteq \ker \varphi$, the kernel of φ.

For the other inclusion we let $w = a_{i_1}^{\epsilon_1} \cdots a_{i_l}^{\epsilon_l} \in \ker \varphi$. We proceed to show that $w \in \gamma_{n+1}\gamma_2(F)$. We may assume that $\gamma_{n+1}\gamma_2(F) = \{e\}$ and conclude $w = e$. Since w involves only finitely many generating symbols of F, we may assume that $w \in F_m$ and $\gamma_{n+1}\gamma_2(F_m) = \{1\}$. Since $w \in \ker \varphi$, $\alpha_{ii}(w) = 1$ for all i which implies that $\mathrm{w}(i) = 1$ and hence $w \in \gamma_2(F)$. To complete the proof of the theorem it is enough to show that if for some $s \in \{1,\ldots, n\}$, $w \in \gamma_s\gamma_2(F_m)$ then $w \in \gamma_{s+1}\gamma_2(F_m)$. Indeed if $w \in \gamma_s\gamma_2(F_m)$, then modulo $\gamma_{s+1}\gamma_2(F_m)$,

$$w = b_{s_1}^{r_1} \cdots b_{s_t}^{r_t}, \text{ by Lemma 1.}$$

Since $w \in \ker \varphi$, $\alpha_{s+1,1}(w) = 0$ and by Lemma 4(x) this implies that

$$\begin{aligned} 0 &= \alpha_{s+1,1}(b_{s_1}^{r_1} \cdots b_{s_t}^{r_t}) \\ &= \sum_{i=1}^{t} r_i \alpha_{s+1,1}(b_{si}), \end{aligned}$$

and by Lemma 6, $r_1 = \cdots = r_t = 0$. Thus $w \in \gamma_{s+1}\gamma_2(F_m)$. This completes the proof of the main theorem.

Concluding Remarks

Remark 2. As promised in Remark 1, let us suppose that $b_{n_1}^{r_1} \cdots b_{n_t}^{r_t} = e$ modulo $\gamma_{n+1}\gamma_2(F_m)$. Then we have

$$\begin{aligned} 0 &= \alpha_{n+1,1}(b_{n_1}^{r_1} \cdots b_{n_t}^{r_t}) \\ &= \sum_{r=1}^{t} r_i \alpha_{n+1,1}(b_{n_i}) \qquad \text{(by Lemma 4(x))} \end{aligned}$$

and hence $r_1 = \cdots = r_t = 0$ (by Lemma 6). We record this as

COROLLARY 1. *Modulo $\gamma_{n+1}(\gamma_2(F_m))$, $\gamma_n\gamma_2(F_m)$ is freely generated by B_n.*

Remark 3. By Lemma 4(v) we have $\alpha_{i,i-1}(a_k^{-1}) = -x_{k,i-1}^{-1}x_{k,i}^{-1}\lambda_{i,i-1}^{(k)}$. Using this, the product rule 4(iii) and $0 = \alpha_{i,i-2}(a_k a_k^{-1})$ we can compute $\alpha_{i,i-2}(a_k^{-1})$ and similarly $\alpha_{ij}(a_k^{-1})$ for all $i - j \in \{1,..., n\}$. It follows that if $w = a_{i_1}^{\epsilon_1} \cdots a_{i_l}^{\epsilon_l}$ is an arbitrary word of F then $\langle w \rangle = \langle a_{i_1} \rangle^{\epsilon_1} \cdots \langle a_{i_l} \rangle^{\epsilon_l}$ can be effectively computed. Since $ZG[\Lambda]$ has a solvable word problem, it can be decided whether or not $\langle w \rangle$ is the identity matrix. Thus we have the following,

COROLLARY 2[1]. *$F/\gamma_n\gamma_2(F)$ has solvable word problem.*

Remark 4. Let $\Lambda = \{\lambda_{i,i-1}^{(k)}, n + 1 \geqslant i \geqslant 2, \ k = 1, 2,...\}$ be a set of independent and commuting indeterminates and let $Z[\Lambda]$ be the polynomial ring over Z. Using the theory of basic commutators it can be seen that the multiplicative group of $n + 1 \times n + 1$ matrices over $Z[\Lambda]$ generated by

$$\begin{bmatrix} 1 & & & & \\ \lambda_{21}^{(k)} & 1 & & \bigcirc & \\ 0 & \lambda_{32}^{(k)} & 1 & & \\ \vdots & & & & \\ 0 & \cdots & 0 & \lambda_{n+1,n}^{(k)} & 1 \end{bmatrix}, \quad k = 1, 2,...$$

[1] The referee has pointed out that every finitely generated linear group has solvable word problem (M. O. Rabin, unpublished).

is isomorphic to $F/\gamma_{n+1}(F)$. According to a suggestion of Professor P. Hall (communicated to the second author a few years ago by Dr. H. Heineken) the same proof can be carried over if $Z[\Lambda]$ is replaced by $Z[\lambda]$ and $\lambda_{i,i-1}^{(k)}$ is replaced by $\lambda^{2^{nk+(i-2)}}$.

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